Estimating Black-Scholes

Steven Lillywhite *

July 31, 2011

Abstract

We compare estimating Black-Scholes option prices for GBM by estimating parameters from a batch of simulations and using exact formulas versus Monte Carlo pricing directly from the simulations.

1 Introduction

Suppose we are confronted with a batch of simulations of a geometric Brownian motion process(GBM). Without loss of generality, we can assume that all simulations have the same initial point. We shall also assume that the simulations are under a risk-neutral measure. Using these simulations, we could compute the price of an option on the GBM in at least two distinct ways. For the moment, we shall focus on European call or put options. We could estimate parameters using statistics of the simulations, and then compute the exact Black-Scholes option prices using these estimated parameters. Or we could use the simulations directly to compute option prices in the Monte Carlo way as the average of the payoffs over all the paths. One may ask which method would be preferable in general. We shall attempt to answer his question when we know the generating stochastic process is GBM.

In what follows below, we shall at times write some messy formulas. The idea is to have them recorded here so that we can use this document as a guide to code the formulas, without having to rework the calculations.

2 Quick Review of MLE estimation

Let us assume that we have an iid sample $\{Z_i\}$ of size n from a normal distribution $N(\mu, v)$. Let us consider joint estimation of μ and v. We shall denote the vector of parameter values $\theta \stackrel{\text{def}}{=} (\mu, v)$, with $\hat{\theta}$ denoting the vector of their estimates. Then the maximum likelihood estimator (MLE) for the mean is given by the sample mean

*Email: stevenlillywhite2@yahoo.com

$$\hat{\mu}_n \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n}(Z_i) \tag{2.1}$$

and the MLE for the variance is

$$\hat{v}_n \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n}(Z_i^2 - \hat{\mu}) \tag{2.2}$$

Remark 1. *Often one considers the same formula to estimate the variance, but dividing by $n-1$, instead of n. Such an estimator is unbiased, whereas the MLE is not. However, the MLE is asymptotically unbiased, and has lower variance. One could make simple adjustments to the formulas that follow to accomodate the other estimator.*

Standard asymptotic theory for MLE gives

$$\sqrt{n}(\hat{\theta} - \theta) \to N(0, I^{-1}) \tag{2.3}$$

where I is the Fisher Information matrix (of one of the samples). In this case, the Fisher Information matrix is

$$I = \begin{pmatrix} \frac{1}{v} & 0 \\ 0 & \frac{1}{2v^2} \end{pmatrix} \tag{2.4}$$

2.1 Delta Method

Suppose we have a function $g(x, y)$ which is differentiable in each variable. Then the so-called "delta method", which may be proved using a Taylor expansion of g, implies that

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \to N(0, \nabla g(\theta)^T I^{-1} \nabla g(\theta)) \tag{2.5}$$

2.2 Change of Variable

The MLE is well-behaved under a change of variable of the parameter vector θ. Let $h(x, y)$ be an injective function. Then the MLE of $h(\theta)$ is $h(\hat{\theta})$. Furthermore, the Fisher Information matrix transforms in a simple way when h is differentiable. Let $\beta \stackrel{\text{def}}{=} h(\theta)$. Then

$$I(\theta) = J^T I(\beta) J \tag{2.6}$$

where J is defined as $J_{i,j} \stackrel{\text{def}}{=} \frac{\partial \beta_i}{\partial \theta_j}$.

2.3 Geometric Brownian Motion Estimation

We shall consider samples from a GBM process which we shall write as

$$dX = \mu X dt + \sigma X dW \tag{2.7}$$

Let us suppose we have a batch of m independent simulations for the process (2.7) as above, and that we wish to estimate the vector of parameters $\theta \stackrel{\text{def}}{=} (\mu, \sigma)$. If we make the change of variable $Y = log(X)$ and apply Itô's lemma, we get

$$dY = (\mu - \tfrac{\sigma^2}{2})dt + \sigma dW \tag{2.8}$$

From which we derive

$$Y(t) = Y(0) + (\mu - \tfrac{\sigma^2}{2})t + \sigma W(t) \tag{2.9}$$

Although not necessary, for ease of presentation we shall assume that the total time T is divided into n equal increments of Δt, so $T = n\Delta t$. Let $Y_i \stackrel{\text{def}}{=} Y(i\Delta t)$ and $Z_i \stackrel{\text{def}}{=} Y_i - Y_{i-1}$. Hence the Z_i are n independent draws from $N((\mu - \frac{\sigma^2}{2})\Delta t, \sigma^2 \Delta t)$.

Since our ultimate goal is to estimate the error in Black-Scholes prices, we could work with the variables $(\mu - \frac{\sigma^2}{2})\Delta t$ and $\sigma^2 \Delta t$ directly. However, as a point of general interest, we'll examine the uncertainty in estimating the GBM parameters μ and σ directly.

Let us consider then the change of variable mapping θ to β (in the notation above)

$$(\mu, \sigma) \mapsto ((\mu - \tfrac{\sigma^2}{2})\Delta t, \sigma^2 \Delta t) \tag{2.10}$$

which is injective on the upper half plane defined by $\sigma > 0$. Then the Fisher Information matrix transforms as $J^T I(\beta) J$ where

$$J = \begin{pmatrix} \Delta t & -\sigma \Delta t \\ 0 & 2\sigma \Delta t \end{pmatrix} \qquad I(\beta) = \begin{pmatrix} \frac{1}{\sigma^2 \Delta t} & 0 \\ 0 & \frac{1}{2\sigma^4 \Delta t^2} \end{pmatrix} \tag{2.11}$$

Working this out, we get

$$I((\mu, \sigma)) = \begin{pmatrix} \frac{\Delta t}{\sigma^2} & -\frac{\Delta t}{\sigma} \\ -\frac{\Delta t}{\sigma} & \Delta t + \frac{2}{\sigma^2} \end{pmatrix} \qquad I^{-1}((\mu, \sigma)) = \begin{pmatrix} \frac{\sigma^2(2+\sigma^2 \Delta t)}{2\Delta t} & \frac{\sigma^3}{2} \\ \frac{\sigma^3}{2} & \frac{\sigma^2}{2} \end{pmatrix} \tag{2.12}$$

We shall summarize the above results in a proposition.

Proposition 1. *Consider a GBM process $dX = \mu X dt + \sigma X dW$. Suppose we have m paths of n equal increments Δt, with the same initial value $X(0)$. We denote the observations as $\{X_{ij}\}$, where $i = 1, 2, \ldots, n$ denotes the values of the jth path ($j = 1, 2, \ldots, m$) at time*

$i\Delta t$. Then the MLE estimators of μ and σ are given by

$$\hat{\sigma} = \sqrt{\hat{v}} \tag{2.13}$$

$$\hat{\mu} \stackrel{def}{=} \frac{1}{mn\Delta t} \sum_{ij} Z_{ij} + \frac{\hat{\sigma}^2}{2} \tag{2.14}$$

$$\hat{v} \stackrel{def}{=} \frac{1}{mn\Delta t} \left[\sum_{ij} (Z_{ij}^2) - \frac{1}{mn} \left(\sum_{ij} Z_{ij} \right)^2 \right] \tag{2.15}$$

and

$$\sqrt{mn}\left[(\hat{\mu}, \hat{\sigma}) - (\mu, \sigma)\right] \to N(0, \Sigma) \qquad \textit{(in distribution)} \tag{2.16}$$

where

$$\Sigma = \begin{pmatrix} \frac{\sigma^2(2+\sigma^2\Delta t)}{2\Delta t} & \frac{\sigma^3}{2} \\ \frac{\sigma^3}{2} & \frac{\sigma^2}{2} \end{pmatrix} \tag{2.17}$$

3 Quick Review of Black-Scholes Formula

Lemma 1. *Let $X(t)$ be a stochastic process which is log-normal at time T. Let $m = E(log(X(T))|\mathcal{F}_t)$ and $v = Var(log(X(T))|\mathcal{F}_t)$. Let $\beta, K_1 \leq K_2 \in \mathbb{R}$. Then*

$$E[X(T)^\beta I(K_1 < X(T) < K_2)|\mathcal{F}_t] = \tag{3.1}$$

$$e^{\beta m + \frac{1}{2}\beta^2 v} \left[N\left(\frac{log(K_2) - m - \beta v}{\sqrt{v}} \right) - N\left(\frac{log(K_1) - m - \beta v}{\sqrt{v}} \right) \right] \tag{3.2}$$

where N is the cumulative standard normal distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz$$

Remark 2. *This formula makes sense for $K_1 = 0$, and also for $K_2 = \infty$.*

Let us suppose that $X(t)$ is the GBM process (2.7) under a risk-neutral measure. Let us fix the initial value $X(0)$, the risk-free rate r(assumed to be constant), the total time, T, and assume that the risk-neutral drift μ is possibly different from the risk-free rate. Then in the formula of Lemma 1, we have $m = log(X(t)) + (\mu - \frac{\sigma^2}{2})T$, and $v = \sigma^2 T$. Let $K > 0$ denote a strike price. Recall that a European call option may be priced via

$$e^{-r(T-t)} E[(X(T) - K)I(K < X(T) < \infty)|\mathcal{F}_t] \tag{3.3}$$

Let us give a name to this random variable

$$\Upsilon \stackrel{\text{def}}{=} e^{-rT}(X(T) - K)I(K < X(T) < \infty) \tag{3.4}$$

So that the Black-Scholes call option price is given by $E[\Upsilon|\mathcal{F}_t]$. For applications below, we shall compute a few statistics of this random variable, both easily evaluated using Lemma 1.

Lemma 2.

$$E[\Upsilon|\mathcal{F}_t] = e^{-r(T-t)}\left[e^{m+\frac{v}{2}}N\left(\frac{m+v-\log K}{\sqrt{v}}\right) - KN\left(\frac{m-\log K}{\sqrt{v}}\right)\right] \tag{3.5}$$

$$Var[\Upsilon] = e^{-2r(T-t)}A - (E[\Upsilon|\mathcal{F}_t])^2 \tag{3.6}$$

where A is defined via

$$A \stackrel{def}{=} e^{2m+2v}N\left[\frac{m-\log K+2v}{\sqrt{v}}\right] - 2Ke^{m+\frac{v}{2}}N\left[\frac{m-\log K+v}{\sqrt{v}}\right] + K^2N\left[\frac{m-\log K}{\sqrt{v}}\right] \tag{3.7}$$

Similar formulas hold for put options.

4 Computing Black-Scholes with Estimated Parameters

Next we shall apply the delta method described above. Let $g(\mu,\sigma)$ be the Black-Scholes call-option price (3.5), considered as a function of μ and σ. We have then

$$g(\mu,\sigma) \stackrel{\text{def}}{=} e^{-rT}\left[X(0)e^{\mu T}N\left(\frac{\log\frac{X(0)}{K}+\mu T+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - KN\left(\frac{\log\frac{X(0)}{K}+\mu T-\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\right] \tag{4.1}$$

Then we can work out that

$$\frac{\partial g}{\partial \mu} = e^{-rT}\left[X(0)Te^{\mu T}N\left(\frac{\log\frac{X(0)}{K}+\mu T+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\right. \tag{4.2}$$

$$\left. + X(0)e^{\mu T}\frac{\partial}{\partial\mu}N\left(\frac{\log\frac{X(0)}{K}+\mu T+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - K\frac{\partial}{\partial\mu}N\left(\frac{\log\frac{X(0)}{K}+\mu T-\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\right] \tag{4.3}$$

$$\frac{\partial g}{\partial \sigma} = e^{-rT}\left[X(0)e^{\mu T}\frac{\partial}{\partial\sigma}N\left(\frac{\log\frac{X(0)}{K}+\mu T+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - K\frac{\partial}{\partial\sigma}N\left(\frac{\log\frac{X(0)}{K}+\mu T-\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\right] \tag{4.4}$$

where

$$\frac{\partial}{\partial\mu}N\left(\frac{\log\frac{X(0)}{K}+\mu T\pm\frac{\sigma^2}{2}T}{\sigma}\right) = \frac{1}{\sqrt{2\pi}}exp\left\{-\frac{(\log\frac{X(0)}{K}+\mu T\pm\frac{\sigma^2}{2}T)^2}{2\sigma^2 T}\right\}\frac{\sqrt{T}}{\sigma} \tag{4.5}$$

$$\frac{\partial}{\partial\sigma}N\left(\frac{\log\frac{X(0)}{K}+\mu T\pm\frac{\sigma^2}{2}T}{\sigma}\right) = -\frac{1}{\sqrt{2\pi}}exp\left\{-\frac{(\log\frac{X(0)}{K}+\mu T\pm\frac{\sigma^2}{2}T)^2}{2\sigma^2 T}\right\}\left[\frac{\log\frac{X(0)}{K}+\mu T}{\sigma^2\sqrt{T}}\mp\frac{\sqrt{T}}{2}\right] \tag{4.6}$$

From this, it is straightforward to see that $g'(\sigma) \neq 0$. Applying the delta method described above, we obtain the error in the Black-Scholes formula using estimated parameters $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$.

$$\sqrt{mn}\big(g(\hat{\mu}, \hat{\sigma}) - g(\mu, \sigma)\big) \to N(0, w) \qquad \text{(in distribution)} \tag{4.7}$$

where w is given by

$$w = \nabla g(\theta)^T \Sigma \nabla g(\theta) \tag{4.8}$$

where $\nabla g(\theta) = (\frac{\partial g}{\partial \mu}, \frac{\partial g}{\partial \sigma})$ and Σ is given by (2.17).

5 Computing Black-Scholes with Monte Carlo

This case is much simpler to analyze. Let us consider the random variable (3.4)

$$\Upsilon \stackrel{\text{def}}{=} e^{-rT}(X(T) - K)I(K < X(T) < \infty) \tag{5.1}$$

where X is the GBM process (2.7) as above. The mean of this random variable is the Black-Scholes option price. We can estimate the mean of a random variable with the sample mean. For our batch of independent simulations of X, we compute the average of this call formula for the final path values. Let us denote this (for m simulations)

$$\Upsilon_m \stackrel{\text{def}}{=} \frac{1}{m} \sum_{i=1}^{m} e^{-rT}(X_i(T) - K)I(K < X_i(T) < \infty) \tag{5.2}$$

By the central limit theorem, we get

$$\sqrt{m}(\Upsilon_m - E[\Upsilon]) \to N(0, Var[\Upsilon]) \tag{5.3}$$

and we have computed both the mean and variance of Υ above in Lemma 2.

6 Summary

We have considered a scenario involving m simulations of independent GBM simulations with n increments each. The process is assumed to be under a risk-neutral measure, and all simulations have the same initial point. We considered computing a Black-Scholes European call option price using the information in the simulations by either estimating GBM parameters and plugging the estimates into the closed-form Black-Scholes formula, or by computing the Monte Carlo price directly out of the simulations. In both cases, we derived asymptotic normal densities for the error in the Black-Scholes price, where by error we mean the difference between the actual price using the known parameters which generated the simulations and the option price obtained by estimation techniques. We have double-checked all formulas using monte carlo testing.

7 Some Numbers

7.1 An Example

```
option type: GBM CALL ; riskNeutralDrift: 0.05 ; vol: 0.2 ; int rate: 0.05
init price: 30.0 ; strike: 100.0
timeToMat: 10.0 ; dt: 1.0
num time steps for each path: 10
num simulations of paths: 2000
theoretical option price with known parameters: 1.745647
BS formula with known drift and estimated volatility asymptotic error stddev: 0.027552
BS formula with estimated drift and volatility asymptotic error stddev: 0.098865
Monte Carlo option pricing asymptotic error stddev: 0.209262
```

7.2 Same Example Equal Number Data Points For Both Estimations

```
option type: GBM CALL ; riskNeutralDrift: 0.05 ; vol: 0.2 ; int rate: 0.05
init price: 30.0 ; strike: 100.0
timeToMat: 10.0 ; dt: 10.0
num time steps for each path: 1
num simulations of paths: 2000
theoretical option price with known parameters: 1.745647
BS formula with known drift and estimated volatility asymptotic error stddev: 0.087148
BS formula with estimated drift and volatility asymptotic error stddev: 0.156218
Monte Carlo option pricing asymptotic error stddev: 0.209262
```